Numerical modeling of the mean exchange through the Strait of Gibraltar

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The three-dimensional Princeton Ocean Model is used to investigate the mean flow and the hydraulics regime in the Strait of Gibraltar. The model makes use of a coast-following curvilinear orthogonal grid that includes the Gulf of Cadiz and the Alboran Sea, with very high resolution in the strait (\(\sim\)500 m). A lock-exchange initial condition is used: The western part of the model domain is filled with Atlantic water, whereas the eastern part is filled with Mediterranean water. A conservative, nondiffusive, and nondispersive numerical scheme for the horizontal tracers advection has been implemented to simulate the free-flow adjustment for the lock-exchange initial condition. Predicted velocities, interface depth, water, and salinity transports are comparable with observed data. Model results describe circulation of the Mediterranean outflow in detail and demonstrate the presence of a mixing layer between the Atlantic and Mediterranean water. The hydraulic regime is analyzed, calculating the composite Froude number for three layers within the strait. Results indicate the presence of a hydraulic control at Camarinal Sill and in the northern part of Tarifa Narrows. In order to understand if the Strait of Gibraltar is in the submaximal or maximal regime we have formulated a cross-strait mean composite Froude number. This formulation allowed us to limit the values of the composite Froude number in the range of 1.2–1.5 at Camarinal Sill and between 0.4 and 0.8 at Tarifa Narrows. We have concluded that the mean exchange simulated by the model is in submaximal regime.


1. Introduction

Since ancient times the Fretum Gaditanum (the Roman name of the Strait of Gibraltar) has fascinated and captured mankind’s imagination: For a long time, Hercules’ pillars, the two promontories of Calpe (now Gibraltar) in Spain and Abila (now Jebel Sidi Moussa) in Morocco, were thought to be the extreme edges of the Earth. Nowadays the strait still attracts considerable interest as a relevant factor in the Mediterranean thermohaline circulation and the intermediate and deep water circulation of the North Atlantic [Reid, 1979; Price et al., 1993].

The narrow and shallow Strait of Gibraltar connects the Atlantic Ocean with the Mediterranean Sea. It is \(\sim\)60 km long and \(\sim\)20 km wide, with a minimum width of 15 km near Tarifa and a shallow sill located near Camarinal (west of Tarifa), with a minimum depth of \(<\)300 m (Figure 1).

An excess of evaporation with respect to the freshwater input and the conservation of mass and salt in the Mediterranean Sea drive the mean circulation in the strait. The mean circulation, generally called inverse estuarine, is characterized by two counterflowing currents: In the upper layer, Atlantic water with a salinity of \(\sim\)36.2 practical salinity units (psu) flows eastward, spreading into the Mediterranean Sea, and in the lower layer, Mediterranean water with a salinity of \(\sim\)38.4 psu flows westward toward the Atlantic Ocean [Lacombe and Richez, 1982].

As initially suggested by Bryden and Stommel [1984], the mean circulation of the Strait can be described as a two-layer system hydraulically controlled at Camarinal Sill.

For a steady two-layer nonrotating inviscid flow through a rectangular cross section the hydraulic control occurs when

\[
G^2 \equiv F_1^2 + F_2^2 = 1
\]

with

\[
F_i^2 = \frac{u_i^2}{g h_i},
\]

where \(G\) is the composite Froude number; \(F_i\), \(u_i\), and \(h_i\) represent the layer Froude number, velocity, and thickness of the upper (\(i = 1\)) and lower (\(i = 2\)) layers, respectively; and \(g' = g \Delta \rho / \rho_2\) is the reduced gravity (\(\Delta \rho = \rho_2 - \rho_1\)) for a
rigorous derivation of equation (1) see Armi [1986] and Lawrence [1990]). At the hydraulic control locations the flow undergoes a transition from subcritical \((G^2 < 1)\) to supercritical \((G^2 > 1)\). A hydraulic jump occurs instead when the flow undergoes a transition from supercritical to subcritical. Hydraulic jump is characterized by turbulent dissipation of energy.

[7] Adding to mass and salt conservation the further constraints of a hydraulic control at Camarinal Sill and the overmixing criterion [Stommel and Farmer, 1953] to the Mediterranean Sea, Bryden and Stommel [1984] found that the overmixed limit corresponds to a salinity difference between the Atlantic Ocean and Mediterranean Sea of 1.7 psu, with inflow and outflow transports of \(1.7 \times 10^6\) and \(1.6 \times 10^6\) m\(^3\) s\(^{-1}\), respectively. They also found that the interface depth over Camarinal Sill was located at middepth \((h_1 = h_2)\).

[8] Motivated by the complex geometry of the Strait, Farmer and Armi [1986] considered the case of a mean circulation hydraulically controlled both at Camarinal Sill and at the contraction near the exit of the surface flow into the Mediterranean. Applying these two hydraulic constraints in a steady two-layer nonrotating inviscid flow model, they found that the exchange flow rate through the Strait reaches a maximal value. Differently from Bryden and Stommel [1984], they found that the minimum salinity difference admitted between the Atlantic Ocean and the Mediterranean Sea is \(\sim 2.0\) psu. The maximum exchange for the model of Farmer and Armi is also characterized by a deeper interface between the two layers over Camarinal Sill and a transport 20% lower than those predicted by Bryden and Stommel.

[9] To explain the observed seasonal cycle of the surface inflow, deduced from the sea level difference between Gibraltar and Ceuta, Bormans et al. [1986] found that the exchange rate through the strait is mainly submaximal. That is, the flow through the strait is controlled only over Camarinal Sill.

[10] In order to improve the models of Bryden and Stommel [1984] and Farmer and Armi [1986], other features, such as Earth’s rotation, friction, and nonrectangular cross sections for the configuration of the strait, have been added in literature. Assuming potential vorticity conservation, Bormans and Garrett [1989a] estimated the influence of the Earth’s rotation on the flow. They found that the main effect of the Earth’s rotation is to produce a tilt across the strait of the interface between the two layers, which is minimum at Camarinal Sill section, while it is enhanced to the east of Tarifa Narrows. Also, Dalziel [1990] estimated the influence of the Earth’s rotation on the flow, predicting a decrease in the exchange up to one-third, with respect to the nonrotating case. Including friction in the model of Farmer and Armi [1986], Bormans and Garrett [1989b] estimated the effect of both interfacial and bottom friction. They concluded that using reasonable drag coefficients (compatible with dissipation measurements by Wesson and Gregg [1988]), the interfacial friction has a smaller influence on the exchange than the bottom friction. They also found that by including the bottom friction on the sloping sides of the strait, the maximal

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**Figure 1.** Chart of the Strait of Gibraltar, adapted from Farmer and Armi [1988], showing the principal geographic features referred to in the text. Locations of current meter mooring (solid square) deployed during the Gibraltar Experiment (October 1985 to October 1986) are also shown. Areas deeper than 400 m are shaded.
and submaximal solutions tend to come closer than in the inviscid model. Using more realistic bathymetry modeled via triangular or parabolic cross-strait sections, Bormans and Garrett [1989b] and Dalziel [1988] showed that the exchange is reduced and the interface between the two layers is raised compared with a simple rectangular cross-section bathymetry.

[11] Using a three-dimensional (3-D) general ocean circulation model, Wang [1989] was able to describe most of the aspect of the mean circulation of the Strait of Gibraltar. However, because of the relatively low horizontal and vertical resolutions (≈5 km and 50 m, respectively), the model did not reproduce the hydraulic control over Camarinal Sill. Wang found, in fact, that while the surface flow is supercritical, i.e., hydraulically controlled on the eastern entrance, the bottom flow is subcritical, i.e., not controlled on the western entrance, over Camarinal Sill. To our knowledge, Wang’s model is the only three-dimensional ocean circulation model applied to the Strait of Gibraltar in the last 10 years.

[12] In the present work we describe a numerical investigation of the mean exchange of the Strait of Gibraltar using the Princeton Ocean Model (POM) [Blumberg and Mellor, 1987] in a very high-resolution configuration (Δx, Δy < 500 m within the Strait) in order to represent all dominant topographic features of the Strait well. In particular, the model described in this paper simulates the nature of the mean exchange through the Strait, i.e., whether it is maximal or submaximal, by determining the locations of the hydraulic control.

Figure 2. Orthogonal curvilinear model grid; the calculated maximum departure of the grid cells from a rectangular shape is <10^{-12}.

Figure 3. Model bathymetry; contour interval is 250 m.
The paper is organized as follows: In section 2 the model is described, with particular attention to the improvements we introduced in the original POM code in order to better describe the physics of the Strait. Model results and discussion about the computation of the Froude number are described in section 3. Finally, section 4 contains the summary and conclusion.

2. Model Description

The model used for this study is the three-dimensional, $\sigma$ coordinate POM model developed by Blumberg and Mellor [1983]. It is a free surface (primitive equations) hydrostatic model. It has been extensively used for a wide range of oceanic problems, including estuarine and shelf-circulation studies [Blumberg and Mellor, 1983; Oey et al., 1985], data assimilation in the Gulf Stream [Mellor and Ezer, 1991], and general circulation studies in the Mediterranean Sea [Zavatarelli and Mellor, 1994].

The model numerically solves the momentum equation, the continuity equation, and the tracer (temperature and salinity) equations in finite difference form along with a nonlinear equation of state $\rho = \rho(\theta, S, P)$ [Mellor, 1991b], which couples the two active tracers to the fluid velocity. Vertical turbulent mixing processes are parameterized with the 2.5 order turbulent closure submodel of Mellor and Yamada [1982]. The horizontal momentum, heat, and salt small scale mixing processes are parameterized as horizontal diffusion (along $\sigma$ surfaces), depending on the horizontal velocity shear and on the grid spacing via the Smagorinsky [1963] diffusion scheme:

![Figure 4.](image-url)
where $\Delta x$ and $\Delta y$ are the grid sizes, $C$ is a constant taken to be 0.2 in this study, and $S$ and $N$ are the mean shear and the normal stress, respectively.

\[
\begin{align*}
A_H(x,y,t) = A_\Omega(x,y,t) &= C \Delta x \Delta y \left[ N^2(x,y,t) + S^2(x,y,t) \right].
\end{align*}
\] (3)

The model uses an explicit leapfrog scheme for time stepping, except for the vertical diffusion terms, which are treated with an implicit scheme. To provide free surface variations in explicit differencing, the model also solves, with a small time step, a set of vertically integrated equations of continuity and motion, usually called external mode. For computer time economy the 3-D equations, usually called internal mode, are solved with a larger time step, limited by the Courant-Friedrichs-Lewy condition for the internal gravity wave speed, using a time-splitting technique. The external and internal time steps used in this study are of 1 s and 60 s, respectively.

The region covered by our model includes the Strait of Gibraltar and the two subbasins connected to it, i.e., the Gulf of Cadiz and the Alboran Sea. The horizontal model domain extends longitudinally from 10°W to 4°E and meridionally from 33°N to 39°N and is covered with a coast-following curvilinear orthogonal grid made by 306 by 253 grid points (Figure 2). Note that the resolution is much higher in the strait (~500 m) than in the eastern (8–15 km) and western ends (10–20 km), so that the dynamics in the strait will be well resolved. The vertical model grid is made by 25 σ levels. The σ coordinate is defined as $\sigma = (z - \eta)/(H + \eta)$; it ranges from $\sigma = 0$ at $z = \eta$ (at the surface) to $\sigma = -1$ at $z = -H$ (at the bottom). The σ levels follow a logarithmic distribution at the surface in order to achieve a better resolution of the upper layer stratification and a uniform distribution in the rest of the water column.

2.1. Model Grid and Bathymetry

The model topography was constructed by bilinear interpolation of the depth data onto each grid point of the horizontal model grid. The depth data were obtained by merging the high-resolution (<1 km) topographic data set of the Strait of Gibraltar provided by the Laboratoire d’Océanographie Dynamique et de Climatologie with the relatively low-resolution (5 min) U.S. Navy Digital Bathymetric Data Base-5 data set (available from U.S. Naval Oceanographic Office, Bay St. Louis, Mississippi at http://128.160.23.42/dbdbv/dbdbv.html) for the Alboran Sea and the Gulf of Cadiz.

In an attempt to reduce the well-known pressure gradient error produced by σ coordinate grids in regions of steep topography [Haney, 1991], an additional smoothing was applied where the slope ($\delta H/H$) was $>0.2$, as suggested by Mellor et al. [1994]. In order to estimate the residual pressure gradient error, the model was integrated for 1 year without initial horizontal density gradient, i.e., with salinity and temperature fields varying only with depth, with no open boundary applied, i.e., closed domain, and without any other external forcing. The maximum intensity of erroneous currents introduced by the σ coordinates was 2 cm s$^{-1}$. Since the expected baroclinic velocities are up to 1 m s$^{-1}$, this error seems to be tolerable.

The resulting model topography, with the minimum depth of the shelf set to 25 m, is illustrated in Figure 3; in particular, the bottom topography and the computational grid in the region of the strait is shown in Figure 4. Here the dominant topographic features of the strait are clearly recognizable (from west to east): Spartel Sill (Sp); Tangier basin; Camarinal Sill (Cm), with a depth of 284 m; and Tarifa Narrows.

2.2. Boundary and Initial Conditions

In the vicinity of the eastern and western ends of the computational domain, two open boundaries are defined, on which values of velocity, temperature, and salinity must be specified. In order to minimize the contamination of the interior model solution due to wave reflection at the boundaries, Orlanski [1976] radiation conditions were used for the depth-dependent velocity, and a flow relaxation scheme (FRS) [Martinsen and Engedahl, 1987] was used for the depth-integrated velocity. The aim of the FRS is to relax the prognostic barotropic velocity toward a null transport within specified regions, generally called FRS zones, near the open boundaries. If $U_i$ is the prognostic barotropic variable computed in the interior of the model domain, the relaxed variable in the FRS zones is given by

\[
U = \alpha U_0 + (1 - \alpha) U_i,
\] (5)

where $U_0$ represents the null transport and $\alpha$ is the relaxation parameter, varying from 1 at the open boundary to 0 at the inner edge of the FRS zones. The following polynomial representation, proposed by Jensen [1998], was used for $\alpha$:

\[
\alpha(I) = [(1 - q)l(I) + q]^p, \quad I = 1, 2, \ldots, 10.
\] (6)

with $p = 6$ and $q = 0$. Here $l(I)$ is the distance of the $I$th grid element from the open boundary, and 10 is the number of grid elements in the relaxation zones used in the model. A Newtonian restoring was used for the two active tracers, with a damping timescale of 5 days. Temperature and salinity are restored at all vertical levels toward the initial condition in the same FRS zones defined for the barotropic velocity.

The normal velocities are set to zero along coast boundaries. At the bottom, adiabatic boundary conditions are applied to temperature and salinity, while a quadratic bottom friction with a prescribed drag coefficient is applied to the momentum flux. This is calculated by combining the velocity profile with the logarithmic law of the wall:

\[
C_D = \max[2.5 \times 10^{-3}, k^2 \ln(\Delta \theta/\theta_b)],
\] (7)
where $k$ is the Von Karman constant, $z_0$ is the roughness length set to 1 cm, and $\Delta z_0$ is the distance of the deepest velocity grid point from the bottom. Wind and tidal effects, as well as surface fluxes, are not included in our simulation.

[24] The model is started from rest; it is driven only by an initial density difference without any other forcing. To do this we have filled the model with two water masses, horizontally uniform and vertically stratified, separated by an imaginary dam in the middle of the Strait (longitude 5°42'W) that is removed at the initial time. Using this procedure, the transport of the two layers in the Strait is able to freely adjust to the density gradient as it does in nature. This kind of initialization is generally known as lock-exchange initial condition. The standard scheme is conservative but dispersive, creating spurious temperature and salinity in regions with sharp density gradients [Gross et al., 1999]. Thus, in order to reduce the large numerical error, the Smolarkiewicz upstream-corrected scheme [Smolarkiewicz, 1984, 1990] was implemented and used in the model, instead of the standard centered advection scheme. The Smolarkiewicz scheme, which is a flux-corrected upstream scheme characterized by a small implicit diffusion, has already been used in other oceanographic models, giving satisfactory results [see e.g., Wang, 1984, 1989; Speich, 1992; Herbaut, 1996]. The scheme was implemented as an iterative algorithm: In the first iteration a classical upstream scheme is used, while each following “corrective” iteration reapplies the upstream scheme using a specially defined antidiffusive velocity. The number of iterations is optional; each additional iteration increases the solution accuracy as well as the computing time. The number of iterations

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**Figure 5.** Initial conditions; (a) vertical profiles of salinity (solid line) and temperature (dashed line) for the Gulf of Cadiz; (b) vertical profiles of salinity (solid line) and temperature (dashed line) for the Alboran Sea.
chosen for this study is three, which gives sufficient accuracy for our purposes.

[26] Initial temperature and salinity fields for the Alboran basin have been obtained by a horizontal average, over the Alboran Sea, of the spring Mediterranean Oceanic Data Base data (available at http://modb.oce.ulg.ac.be/modb/welcome.html). Initial fields for the western basin have been obtained by horizontally averaging, over the Gulf of Cadiz,

**Figure 6.** Surface and bottom distribution of salinity after (a and c) 3 hours and (b and d) 15 hours. Contour intervals are (Figures 6a and 6b) 0.25 practical salinity units (psu) and (Figures 6c and 6d) 0.5 psu.

**Figure 7.** The temporal evolution of the along-strait density difference between the extreme edges of the strait.
the temperature and salinity of the spring Levitus [1982] data. In Figure 5 the initial temperature and salinity profiles for the Alboran basin and the Gulf of Cadiz, respectively, are shown.

3. Model Results

3.1. Spin-Up Phase

[27] Immediately after the “dam” is lifted, a sharp cross-strait density discontinuity of 1.4 \( \sigma_0 \) is created between the heavier Mediterranean water and the lighter Atlantic water. The generated along-strait pressure gradient drives the initial motion of the two water masses: Atlantic and Mediterranean waters spread outward, in opposite directions and in geostrophic balance, over the whole dam width, in agreement with Wang’s [1985] model.

[28] The first internal Rossby radius (\( R \)) within the Strait, in our model, is \( \sim 14 \) km. It was calculated using \( R = \sqrt{g' h_e/f} \) [LeBlond and Mysak, 1978], where \( g' \) is the reduced gravity, \( f \) is the Coriolis parameter corresponding to \( 36^\circ \text{N} \), and \( h_e \) is the equivalent depth defined as \( h_e = h_1 h_2 / (h_1 + h_2) \), where \( h_1 \) and \( h_2 \) are the thicknesses of the surface Atlantic inflow and of the deep Mediterranean outflow, respectively. Thus, since the width of the Strait is of the same order of the first internal Rossby radius, the two flows are quickly trapped by the coast. At the coast there is no upstream component of the Coriolis acceleration. As a consequence, two superimposed density currents begin to form and propagate, with a frontal speed of 57 cm s\(^{-1}\) and

Figure 8. (top) The temporal evolution of the eastward (inflow) and westward (outflow) transport. (bottom) The temporal evolution of the outflow salinity transport. Both curves are computed at section C.
69 cm s\(^{-1}\) for the Atlantic and Mediterranean waters, respectively (Figure 6). The lower Mediterranean gravity current reaches the western Strait opening after 6 hours, while the surface Atlantic gravity current reaches the eastern Strait after 15 hours.

3.2. Mean Flow

[29] In the first 3 days of integration the abrupt along-strait vertically integrated density difference (\(\Delta \rho = 1.4 \sigma_0\)) between the extreme edges of the strait generates a baroclinic transport of \(~0.8\) Sv in both layers. In the following 6 months the transport decreases as the along-strait density difference becomes smoother (Figure 7). After 200 days of integration the along-strait density difference reaches a quasi-steady value of 0.99 \(\sigma_0\) that generates a quasi-steady mean transport in both layers and an outflow salinity transport of 0.72 Sv and 1.39 \(\times\) \(10^3\) m\(^3\)s\(^{-1}\), respectively (Figure 8). The inflow \((Q_I)\) and outflow \((Q_M)\) transports are computed, integrating the along-strait velocity vertically from the bottom up to the surface and then laterally across section C (refer to model grid in Figure 4):

\[
Q_I(t) = \int_C ds \int_{\sigma = -1}^{\sigma = 0} u_{in}(z, t) dz
\]

\[Q_M(t) = \int_C ds \int_{\sigma = -1}^{\sigma = 0} u_{out}(z, t) dz,\] (8)

while the outflow salinity transport (OST) is computed by an analogous integration of the along-strait outflow velocity times the salinity excess above a mean salinity of Atlantic inflow of 36.1 psu:

\[
OST(t) = \int_C ds \int_{\Delta S = -1}^{\Delta S = 0} u_{out}(z, t)(S(z, t) - 36.1\text{psu}) dz.
\] (9)

The values of the simulated transports are in good agreement with recent estimates on the direct measurements carried out by Bryden et al. [1994] and Tsimplis and Bryden [2000].

[30] The model also reproduces the two-layer character of the flow within the Strait, as shown in the scatter plot of along-strait steady velocity versus salinity in Figure 9a. Here there is a strong inflow of Atlantic water with salinities <36.5 psu and a strong outflow of Mediterranean water with salinities >38.0 psu. For salinities ranging between 37.0 and 37.5 psu there are both inflow and outflow. In Figure 9b the along-strait mean velocity profile versus salinity is shown,

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**Figure 9.** (a) Scatter plot of velocity versus salinity for the strait of Gibraltar. (b) Profile of the along-strait average velocity versus salinity. Velocities are averaged over salinity layers of thickness \(\Delta S = 0.2\) psu.

**Figure 10.** Comparison between observed and simulated mean current for eight current meter moorings of the Gibraltar Experiment. Observed data are from Candela et al. [1990].
obtained by averaging the velocities over salinity layers of thickness $\Delta S = 0.2$ psu. For salinities $<37.25$ psu there is an inflow velocity that reaches a maximum value of 40 cm s$^{-1}$ at a salinity of 36.25 psu. For salinities $>37.25$ psu there is an outflow velocity that reaches a maximum value of 35 cm s$^{-1}$ at a salinity of 38.4 psu. The transition between inflow and outflow then occurs at a salinity of 37.25 psu, which will be hereinafter taken to define the interfacial isohaline between the Atlantic and Mediterranean waters. Such definition is in agreement with the choice by Lacombe and Richez [1982], Bormans and Garrett [1989b], and Bryden et al. [1994].

In order to quantitatively compare the model results with observed data we have made a linear regression (Figure 10) comparing predicted and observed mean velocity profiles for eight current meter moorings of the Gibraltar Experiment, a large-scale field study carried out for a 1 year period between October 1985 and October 1986 (Figure 1). The mean velocity data are taken from Candela et al. [1990]. The mean errors obtained for the upper and lower layer are 2.1 and 3.8 cm s$^{-1}$, respectively.

Figure 11 shows the salinity structure for the cross-strait sections near Spartel Sill (Figure 11a), Camarinal Sill (Figure 11b), and Tarifa Narrows (Figures 11c and 11d), respectively, after 360 days of integration (hereinafter, the model’s outputs will be always referred to after 360 days of integration). It is evident that in all cross sections the light Atlantic water tends to pile up against the south shore, increasing the thickness of the eastward upper layer flow toward the south, while the heavy Mediterranean water tends to pile up against the north shore, increasing the thickness of the westward lower-layer flow toward the north. At section A the depth of the 37.25 psu isohaline varies from 165 m on the southern side to 122 m on the northern side, with a cross-strait slope of $1.7 \times 10^{-3}$. Near Camarinal Sill the cross section slope of the interface increases up to $4.7 \times 10^{-3}$, while along Tarifa Narrows the cross-strait slope of the interface decreases down to $2.8 \times 10^{-3}$. The predicted value of the cross-strait slope near Camarinal Sill is in agreement with the observed value of $4.6 \times 10^{-3}$ given by Bryden et al. [1994]. In contrast with Bormans and Garrett [1989a] the rotational effect seems to be smaller at the Tarifa Narrows and Spartel Sill sections than at the Camarinal Sill section.

Figure 12 shows the predicted along-strait velocity distribution along the cross-strait sections A, B, C, and D. Cross-strait shear of the along-strait velocity is evident at Tarifa Narrows, where a surface jet is located on the north
Figure 12. Velocity cross-strait sections at (a) Spartel Sill, (b) Camarinal Sill, and (c–d) Tarifa Narrow. Contour intervals are 10 m s$^{-1}$ for the upper layer and 20 m s$^{-1}$ (Figures 12a and 12b) and 10 m s$^{-1}$ (Figures 12c and 12d) for the lower layer.

Figure 13. Contour of potential vorticity multiplied by 10$^6$, with 0.4 m s$^{-1}$ contour interval. Gray level indicate upper layer velocity.
side of the strait. Such cross-strait distribution of the surface velocity is in agreement with the hypothesis of constant potential vorticity (PV) formulated by Bormans and Garrett [1989a]. In fact, as shown in Figure 13, the isolines of constant PV for the upper layer coincide, in the northern side of the Strait, with the streamlines. Thus the upper layer flow, moving eastward, has to compensate the reduction of thickness by decreasing the absolute vorticity \( (f - (\partial u/\partial y)) \) and therefore increasing the northward gradient velocity \( (\partial u/\partial y) \). Nevertheless, it is interesting to note that the region where trajectories and isolines of PV coincide starts just to the west of Camarinal Sill and not from an Atlantic reservoir located to the west of Spartel Sill as suggested by Bormans and Garrett [1989a].

[34] The Mediterranean outflow is westward everywhere, with velocity gradually increasing from Tarifa Narrows to Camarinal Sill. A sketch of the simulated Mediterranean outflow is shown in Figure 14: The region of the Strait delimited by thick lines corresponds to denser fluid with a westward speed \( >20 \text{ cm s}^{-1} \), whereas lines with arrows indicates the paths of jet. This picture illustrates the formation of a westward jet, at the slope beginning east of Camarinal Sill. When the jet arrives in the middle of the Sill, it splits in two parts, with the strong branch in the north. Subsequently, the flow reaches the Tangier basin and the northern branch of the jet vanishes, whereas the other part intensifies to over 100 cm s\(^{-1}\), creating a deep jet that intrudes in the Gulf of Cadiz.

[35] The Mediterranean flux \( (h_2u_2\Delta y) \) through the three cross sections B\(_1\), B\(_2\), and B\(_3\), located between Tarifa Narrows and Camarinal Sill (refer to the Strait map in Figure 14), shows maximal value in the middle of channel, while the along-strait velocity shows the growing of a jet to the east of Camarinal Sill, near the southern side (Figure 15).

[36] In our simulation the PV for the Mediterranean layer is conserved only in regions where bathymetric slope is moderate. For this reason we believe that the formation of the deep westward jet must be due to other causes. Thus, to better analyze the mechanism of lower layer jet formation between Tarifa and Camarinal Sill, we examined the interface depth, pressure, and density at 200 m depth along the longitudinal sections E\(_1\), E\(_2\), and E\(_3\) (Figure 16). We have

**Figure 14.** Sketch of the Mediterranean outflow over Camarinal Sill. The white thick lines delimit outflow velocity \( >20 \text{ cm s}^{-1} \). White arrow represents the Mediterranean jet path.
previously observed that the interface slope increases moving from Tarifa to Camarinal; in particular, Figure 16a shows that the interface depth along the southern sections (E3 and E2) rises nonlinearly from cross section B3 to B1, and Figure 16b indicates a sensible density decrease along the southern section E3. Both these features produce a decrease of pressure (Figure 16c) more evident along the southern section (E3) than in the central part. Therefore a northward positive pressure gradient is generated, with a maximum just at Camarinal Sill. Consequently, the along-strait velocity must increase in the westward direction to balance, at least partially, the northward pressure gradient along the south side, causing the jet formation. In Figure 17a we plot the along-strait lower-layer velocity at 200 m depth for three cross sections: one at Camarinal Sill and the others shifted by three grid points to the west and the east of the sill, respectively. It is evident that the jet is near the south side to the east of the sill, whereas it moves to the north side to the west of the sill. This behavior is related to the vertical mixing just west of the sill due to the abrupt acceleration of the dense Mediterranean water going down the sill. Here, in fact, vertical transport changes sign in the middle of the Mediterranean layer, with positive values in the upper part and negative values in the lower part (Figure 18b). From Figure 17b it is possible to note that in the east section the salinity is quite uniform, ranging from 38.2 to 38.4 psu, whereas a few kilometers westward it is lower than 37.8 psu, except in the north where it remains unchanged at 38.3 psu. Thus salinity mixing produces a dramatic cross-section density gradient.
inducing a maximum southward pressure gradient in the north (Figure 17c). In summary, the maximum pressure gradient shifts toward the north as soon as Mediterranean water flows over Camarinal Sill, inducing a northward shift of the jet.

[37] In the Tangier basin the salinity is more homogeneous along cross-strait sections but presents maximal values in the deeper central part of the strait, constraining the jet in this zone.

3.3. Rotational Effects and Stationary Flow

[38] Some authors have speculated that the effect of Earth’s rotation can be evaluated, assuming a geostrophic balance for the cross-strait momentum equation. In order to

Figure 16. Along-strait sections E₁ E₂ E₃ of (a) interface depth, (b) lower-layer density subtracted by $\rho_0 = 1028$, and (c) pressure subtracted by $P_0 = 2.014 \times 10^6$ at 200 m depth. The vertical dotted line represents the position of cross-strait sections B₁, B₂, and B₃.

Figure 17. Cross section at 200 m depth for (a) velocity, (b) salinity, and (c) northward pressure gradient. Line with triangular mark is three grid points west of Camarinal Sill, square mark is at Camarinal, and circular mark is three grid points east of Camarinal.
better understand the rotational effect within the strait, we have evaluated 10-day averages of all the terms present in the momentum equation. In Figure 18 the vertical profiles of pressure gradient, horizontal advection, vertical transport, Coriolis term, and vertical diffusion in four points within the strait along section E are shown. For the point just west of Camarinal Sill the geostrophic balance is present only for the upper layer, whereas the advection terms are not present.

Figure 18. Vertical profile of terms in the cross-strait momentum tendency equation for a point located along section E (a) near the western entrance of the strait, (b) just west of Camarinal Sill, (c) at Tarifa, and (d) at Gibraltar.
negligible for the Mediterranean layer. Always in this point, friction is ~30% of the maximum forcing only close to the bottom, whereas vertical diffusion contributes to the balance only in the interface layer. Mixing is also particularly active in the interface in the northern part of Tarifa Narrows. For the points near the western entrance of the Strait just east of Tarifa and near Gibraltar there is geostrophic balance in part of the lower layer but not in the upper one, where the advection terms are not negligible.

[39] The 10-day mean tendency of the cross-strait velocity component vanishes everywhere, but looking at its root mean square, it is possible to note that only in regions where geostrophic balance holds is there a perfect stationary flow. Where the advection terms are important, the mean quadratic tendency of the cross-strait velocity is not vanishing and has the same magnitude of the forcing terms at the interface. This indicates that in such regions the flow is subjected to short period oscillations that are canceled when applying a time mean operator of few days.

3.4. Hydraulic Control

[40] To allow a comparison between the model results and the two-layer hydraulic theory, it is necessary to define an interface to discriminate between the Mediterranean and the Atlantic layers. As mentioned in section 3.2., we choose the 37.25 psu isohaline to define such an interface. Figure 19 shows the predicted steady interface depth within the Strait of Gibraltar. Note that the interface depth increases toward the east and south, with an abrupt change near Camarinal Sill.

[41] In order to compute the layer Froude numbers ($F_i$), the interface depth is taken as the height of the upper layer $h_1$, and the lower layer thickness $h_2$ is calculated as $H - h_1$, where $H$ is the water depth. Layer velocities and densities

Figure 19. Two-layer interface depth with 10 m contour intervals. Gray levels indicate bathymetry.

Figure 20. Two-layers composite Froude number (gray levels), calculated using mean layer velocities. Contour lines represent the bathymetry.
(\bar{u}, \bar{v}) are computed by averaging the along-strait velocity and density from the surface to the interface depth for the upper layer and from the interface depth to the bottom for the lower layer. The resulting value of \( G^2 \) is shown in Figure 20. Note that within the Strait \( G^2 < 1 \) everywhere, i.e., the flow is subcritical. However, this result seems to be inconsistent with the simulated character of the flow; in fact, the predicted salinity distribution along the longitudinal axis of the Strait (section E), shown in Figure 21, suggests the presence of a hydraulic jump west of Camarinal Sill. It seems that the direct application of the two-layer hydraulic theory to the Strait of Gibraltar is not obvious, as also argued by Winters and Seim [2000]. As observed by Bray et al. [1995], entrainment and mixing between the Atlantic upper layer and the Mediterranean lower layer are responsible for the creation of a third interfacial layer. Thus, in order to verify the presence of an interface layer, a quantitative method similar to that described by Bray et al. was used. In particular, Bray et al. fitted the salinity profiles with three different straight lines for the upper (\( \ell_u \)), interface (\( \ell_i \)), and lower layers (\( \ell_b \)), respectively. Intersection of the upper layer line with the interface line represents the depth of the lower boundary of the upper layer (\( H_u \)), while intersection of the bottom layer line with the interface line represents the depth of the upper boundary of the bottom layer (\( H_b \)). Consequently, the thickness of the interface layer is defined as the difference \( H_b - H_u \). However, vertical profiles of salinity often are not well represented by these simple functions because in many points (particularly in the surface layer), salinity does not increase monotonically with depth. To overcome this problem, we have fitted the salinity profiles with a cubic polynomial, conditioning the parameter of the polynomial to guarantee a monotonic increasing of salinity with depth. In Figure 22 the salinity profile, cubic polynomial, and interface layer for a point in the Strait just east of Camarinal Sill are shown. In Figure 22, velocity and density profiles are also shown for the same point. Predicted thickness of the interface layer for the whole Strait is shown in Figure 23. It is interesting to note that the thickness has the greatest values just west of Camarinal Sill, indicating the presence of mixing in that region. The thickness of the interfacial layer obtained is in good agreement with that calculated by Bray et al. [1995]. Moreover, in agreement with Bray et al., our simulation

Figure 21. Simulated salinity distribution along the longitudinal axis of the Strait (sec. E).

Figure 22. (a) Salinity, (b) velocity, and (c) density profiles for a point in the strait just east of Camarinal Sill. In Figure 22a are also plotted the monotonic cubic polynomial (square) of salinity profile (circle) and the tangent lines (\( \ell_u, \ell_i, \text{and } \ell_b \)) to the monotonic cubic polynomial.
shows a difference in horizontal transport along the Strait (Figure 24) due to vertical exchange between Atlantic and Mediterranean layers. It is interesting to note that the largest changes occur in the region west of Camarinal Sill (grid point 104), where mixing effects are particularly strong.

To evaluate the effect of more than two layers, it is necessary to extend the usual hydraulic control theory to a multi-layer flow (see Appendix A). In particular, in the presence of a mixed layer we used the multi-layer formulation for three layers with Froude numbers defined as

\[
F_{1}^{2} = \frac{\rho_{1}^{2}}{h_{1}g(1 - r_{1,2})}, \quad F_{2}^{2} = \frac{\rho_{2}^{2}(1 - r_{1,3})}{h_{2}g(1 - r_{1,2})(1 - r_{2,3})},
\]

\[
F_{3}^{2} = \frac{\rho_{3}^{2}}{h_{3}g(1 - r_{2,3})},
\]

where \(r_{ij}\) is the density ratio \(\rho_i/\rho_j\). The composite Froude number is given, to a good approximation, by

\[
G^{2} = F_{1}^{2} + F_{2}^{2} + F_{3}^{2}.
\]
The predicted composite Froude number within the strait is shown in Figure 25. Critical values of $G^2$ can be observed over the whole cross-strait section west of Camarinal Sill and in the northeastern part of Tarifa Narrows. The contribution from the interfacial Froude number ($F^2_2$) is negligible everywhere, whereas the upper value ($F^2_1$) is essential only for the Tarifa Narrow region.

To evaluate the sensitivity of the composite Froude number to the position of the upper and lower interfaces of the mixed layer, we varied (as given by Pratt et al. [1999]) the position of the interfaces of ±10 m about the predicted values; in particular we considered an error of ~25% (±20 m) in determining the thickness of the mixing layer (Figure 26). From the results it is evident that in spite of the introduced errors, the composite Froude numbers are qualitatively

Figure 25. Three-layer composite Froude number (gray levels), calculated using mean layer velocities. Contour lines represents (thin) the bathymetry, (bold) limit the region where $G^2 > 1$.

Figure 26. Composite Froude number for the three-layer case computed along section E using the (solid line) predicted mixed layer thickness (dashed line), increased (+20 m) mixed layer thickness, and (dotted-dashed line) the decreased (−20 m) mixed layer thickness.
similar in the sense that there is a supercritical regime at Camarinal Sill and a subcritical regime at Tarifa Narrows along the section E.

Thus model results shown in Figure 25 seem to indicate the presence of hydraulic control at Camarinal Sill, whereas there is not a clear indication for Tarifa Narrows. The uncertainties come from the application of the two-dimensional hydraulic control theory to a three-dimensional model. In an attempt to eliminate these uncertainties we calculated the cross-strait integrated expression of the Bernoulli Potential $B_c$. For example, in the two-layer formulation the first layer integrated potential is

$$\int_{y_N}^{y_S} B_c(x, y) dy = \int_{y_N}^{y_S} \left( \frac{\pi_1(x, y)}{2} \right) dy + g \int_{y_N}^{y_S} \left( \pi_1(x, y) \pi_1(x) \right) dy,$$

where $y_S$ is the bottom depth and $y_N$ and $y_S$ are the north and south boundary, respectively (note that the overbar indicates vertical average). To apply hydraulic theory, it is necessary to express the Bernoulli Potential and the volume flow rate $Q$ in terms of cross-strait integrated quantities, $(\int_{y_N}^{y_S} h_1(x) dy \equiv h_1(x))$, with the introduction of some coefficients:

$$\int_{y_N}^{y_S} B_c(x) dy = 0.5 C_{ui} \left\{ \bar{\pi}_1(x) \right\} \left\{ \bar{\pi}_1(x) \right\} + g C_{h11} \left\{ \bar{\pi}_1(x) \right\} \left\{ h_1(x) \right\}$$

$$+ g \left\{ \bar{\pi}_1(x) \right\} \left\{ C_{h12} \left\{ h_2(x) \right\} + C_{h1S} \left\{ h_S(x) \right\} \right\} \left\{ \bar{\pi}_1(x) \right\} \left\{ \bar{\pi}_1(x) \right\}$$

$$Q_1 = \int_{y_N}^{y_S} h_1(x) \pi_1(x) dy = C_{q1} \left\{ h_1(x) \right\} \left\{ \bar{\pi}_1(x) \right\} \left\{ \bar{\pi}_1(x) \right\}$$

with

$$C_{ui} = \left\{ \bar{\pi}_1(x) \right\} \left\{ \bar{\pi}_1(x) \right\} C_{h11} = \left\{ \bar{\pi}_1(x) \right\} \left\{ h_1(x) \right\} C_{h12} = \left\{ \bar{\pi}_1(x) \right\} \left\{ h_2(x) \right\}$$

$$C_{q1} = \left\{ h_1(x) \right\} \left\{ h_1(x) \right\} C_{h1S} = \left\{ \bar{\pi}_1(x) \right\} \left\{ h_S(x) \right\}.$$

Equations and coefficients for the second layer can be easily deduced. Coefficients $C_{ui}$, $C_{h11}$, $C_{h12}$, $C_{q1}$, and $C_{h1S}$ (and the same for the second layer) are not constant along the strait, but if we define two smoothed functions for each coefficients, representing higher and lower values at every along-strait position, we can neglect their x-derivative. Hence differentiating the expressions (equations (13) and (14)) in the along-strait direction for the higher and lower set of coefficients and dividing for the width of the Strait $b(x)$, we obtain the same formulation of Froude numbers in terms of mean cross-strait values of velocity current and layer thickness. For two layers we obtain

$$F_1^2 = \frac{C_{ui} \{ \pi_1 \}^2}{C_{h11} g h_1}, F_2^2 = \frac{C_{ui} \{ \pi_2 \}^2}{C_{h22} g h_2}; \ g_r = \frac{1 - \bar{r}_1 C_{h12}}{\bar{r}_2 C_{h22}}.$$

Indicating higher and lower values of coefficients as max $\langle C \rangle$ and min $\langle C \rangle$, we can limit the values of Froude numbers between

$$\text{max} \langle F^2 \rangle = \frac{\text{max} \langle C_{ui} \rangle \langle \pi_1 \rangle^2}{\text{min} \langle C_{h11} \rangle \text{min} \langle g_r \rangle h_1},$$

$$\text{min} \langle g_r \rangle = g \left( 1 - \bar{r}_1 \text{max} \langle C_{h12} \rangle \right) \bar{r}_2 \text{min} \langle C_{h22} \rangle.$$
horizontal tracers. Therefore we have implemented and used the modelers. Previously, this problem was examined by Wang [1984], with a three-dimensional z-level model having a horizontal domain limited only to the strait region, with horizontal and vertical resolution of $[1989]$, with a three-dimensional z-level model having a resolution of the mean circulation, but the model failed to reproduce the results obtained by our numerical model is very detailed, showing all the aspects of hydraulic control at Camarinal Sill, whereas there is not a clear indication for Tarifa Narrows. To remove this uncertainty we formulated cross-strait mean equations in terms of cross-strait mean quantity and confined the solution between two upper and lower limit solutions. This procedure allowed us to give upper and lower limits for the values of the composite Froude number for the cross-strait mean flow. In particular, at Camarinal Sill the mean cross-strait composite Froude number is found in the range of 1.2–1.5, whereas at Tarifa Narrows it was between 0.4 and 0.8. This result allows us to conclude that the mean circulation simulated by the model is in a submaximal regime.

4. Summary and Conclusions

The simulation of the mean exchange through the Strait of Gibraltar is a fascinating challenge for ocean modelers. Previously, this problem was examined by Wang [1989], with a three-dimensional z-level model having a horizontal domain limited only to the strait region, with horizontal and vertical resolution of $\sim 5$ km and 50 m, respectively. Wang was able to describe most of the aspects of the mean circulation, but the model failed to reproduce the hydraulic conditions within the strait.

The coast-following curvilinear grid and terrain-following vertical grid represent indispensable features that greatly improve the ability to reproduce the strait circulation. For these reasons we choose the three-dimensional $\sigma$ coordinate Princeton Ocean Model (POM) to study the mean exchange.

Since the transport of salinity and buoyancy is the principal mechanism involved in the steady strait circulation, it is crucial to use a conservative, nondiffusive, and non dispersive numerical scheme for the advection of horizontal tracers. Therefore we have implemented and used the Smolarkiewicz [1984] scheme instead of using the standard POM’s scheme.

The real dynamics of the Strait of Gibraltar is influenced by nonstationary barotropic forcing as the astronomical tide [Helfrich, 1995] and by the evolution of the mean atmospheric pressure on the Atlantic Ocean and the Mediterranean basin [Harzallah et al., 1993]. However, the first purpose of this study was to demonstrate the ability of our model to reproduce the mean exchange and the hydraulic characteristics of the Strait circulation. The results obtained for integral quantities like the mean transport (0.72 Sv) and the mean outflow salinity transport (1.39 $\times 10^7$ m$^3$ s$^{-1}$), as well as the depth and shape of interface, are in agreement with the actual estimates deduced from measurements and reported in literature. Moreover, the simulated mean currents are in agreement with the available measurements obtained from the large-scale Gibraltar Experiment. The circulation described by our numerical model is very detailed, showing previously unknown features of the mean Strait circulation, but the lack of observations cannot help us to confirm or refute them. In particular, the results for the Mediterranean layer indicate the formation of a jet that is located east of Camarinal Sill, near the south side, whereas westward of the sill, the flow is subject to a hydraulic jump and the jet is flattened and attached on the north border.

\[
\min(F^2) = \frac{\min(C_{\text{min}}) \left(\pi_i \right)^2}{\max(C_{\text{max}}) \max(g_r) h_i^2},
\]

\[
\max(g_r) = g \left(1 - \frac{\min(C_{\text{min}})}{\max(C_{\text{max}})}\right).
\]

An analogue formulation can be found for three layers, and the resulting maximum and minimum composite Froude numbers are plotted in Figure 27. It is interesting to note that significant difference between the two curves is present only at Camarinal Sill and at the beginning of Tarifa Narrows. However, the lower curve overtakes critical values only at Camarinal Sill, whereas at Tarifa Narrows not even the upper limit is $>0.8$.

Appendix A: Hydraulic Control for Multilayer Flow

The motion of a $n$-layer flow is governed by continuity and Bernoulli equation applied to each layer [Baines, 1988]:

\[
h_k \frac{\partial u_k}{\partial x} + u_k \frac{\partial h_k}{\partial x} = Q_k \frac{1}{h_k(\delta)} \frac{\partial}{\partial x} \left(\frac{1}{h_k(\delta)}\right) \quad k = 1, 2, 3, \ldots n
\]

\[
u_{k}\frac{\partial u_{k}}{\partial x} + \sum_{j=1}^{k} \nu_{j} \frac{\partial h_{j}}{\partial x} + \sum_{j=k+1}^{n} \frac{\partial h_{j}}{\partial x} = -g \frac{\partial h_{k}}{\partial x} \quad k = 1, 2, 3, \ldots n
\]

where $u_k$, $h_k$, $Q_k$, $\nu_k$, and $b_k$ are the velocity, thickness, volumetric flow rate, density, and width of the $k$th layer, respectively. In analogy with the treatment of Armi [1986] for a two-layer flow the above quasi-linear system of differential equations can be written in the general form

\[
C \vec{v} = \vec{F}
\]

where $\vec{v}$ and $C$ are defined as

\[
\nu_j = \begin{cases} \frac{\partial h_j}{\partial x} & 1 \leq j \leq n \\ \frac{\partial h_n}{\partial x} & n + 1 \leq j \leq 2n \end{cases},
\]

\[
C = \begin{bmatrix} U & G \\ H & U \end{bmatrix}
\]
In particular, $\mathbf{U}$, $\mathbf{H}$, and $\mathbf{G}$ are defined as

$$
\mathbf{U} = \begin{bmatrix}
  u_1 & 0 & \ldots & 0 \\
  0 & u_2 & \ldots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \ldots & u_n 
\end{bmatrix},
\mathbf{H} = \begin{bmatrix}
  h_1 & 0 & \ldots & 0 \\
  0 & h_2 & \ldots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \ldots & h_n 
\end{bmatrix},
\mathbf{G} = \begin{bmatrix}
  g & g & \ldots & g \\
  g_1 & g & \ldots & g \\
  \vdots & \vdots & \ddots & \vdots \\
  g_{1n} & g_{2n} & \ldots & g 
\end{bmatrix},
$$

(A5)

where $g_{ij} = g(p_j)/(\partial f)$. [53] When the flow is hydraulically controlled, the $\det(\mathbf{C}) = 0$. A simple way to calculate $\det(\mathbf{C})$ is to gather terms containing the product of all thickness and permuting the $4$th and $(n^*+k)$th rows of $\mathbf{C}$ for terms containing the product of $n - 1$ thickness:

$$
\begin{align*}
| \mathbf{C} | &= (-1)^N \{ h_1 u_2 h_3 \ldots h_n \ | \ G \} - u_1 h_2 \ldots h_n \\
&\quad - h_1 u_2 h_3 \ldots h_n \\
&\quad - h_1 h_2 u_3 \ldots h_n \\
&\quad \cdots
\end{align*}
$$

(A6)

where $\Theta$ contains terms combining two and more layer velocities $u_k$. Previous formulation can also be written in a more compact form:

$$
| \mathbf{C} | = (-1)^n \left\{ \prod_{k=1}^{n} h_k g^N \ | \ | \mathbf{M}_h | - \sum_{k=1}^{n} u_k g^{n-1} \prod_{j=1}^{k-1} h_j \right\} + \Theta.
$$

(A7)

with

$$
M_h(i,j) = \frac{G(i,j)}{g} = \begin{cases} 
  1 & i \leq j \\
  \frac{v_j}{v_i} & 1 > j 
\end{cases}
$$

$$
M_n(i,j) = \begin{cases} 
  M_h(i,j) & i \neq k \\
  u_k & 1 = k \ j = k \\
  0 & i = k \ j \neq k
\end{cases}
$$

Remembering that the determinant of a matrix $\mathbf{S}$ defined as

$$
\mathbf{S} = \begin{bmatrix}
  1 & 1 & 1 & \ldots & 1 \\
  r_{1,2} & 1 & 1 & \ldots & 1 \\
  r_{1,3} & r_{2,3} & 1 & \ldots & 1 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  r_{n-1,n} & r_{n,n} & r_{n-1,n} & \ldots & 1
\end{bmatrix}
$$

(A9)

can be calculated as $\det(\mathbf{S}) = \det(\mathbf{W})$ where $W(i,j) = S(i,j) - 1$, so both $\det(\mathbf{M}_h)$ and $\det(\mathbf{M}_n)$ can be easily calculated:

$$
| \mathbf{M}_h | = \begin{bmatrix}
  1 & 1 & 1 & \ldots & 1 \\
  r_{1,2} & 1 & 1 & \ldots & 1 \\
  r_{1,3} & r_{2,3} & 1 & \ldots & 1 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  r_{n-1,n} & r_{n,n} & r_{n-1,n} & \ldots & 1
\end{bmatrix}
$$

(A10)

$$
| \mathbf{M}_n | = \begin{bmatrix}
  1 & 1 & 1 & \ldots & 1 \\
  r_{1,2} & 1 & 1 & \ldots & 1 \\
  r_{1,3} & r_{2,3} & 1 & \ldots & 1 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  r_{n-1,n} & r_{n,n} & r_{n-1,n} & r_{n,n} & \ldots & 1
\end{bmatrix}
$$

(A11)

Thus it is possible to find that

$$
\begin{align*}
| \mathbf{M}_h | &= (1 - r_{1,2})(1 - r_{2,3})(1 - r_{3,4})(1 - r_{4,5}) \cdots (1 - r_{n-1,n}) \\
| \mathbf{M}_n | &= (1 - r_{1,2})(1 - r_{2,3})(1 - r_{3,4})(1 - r_{4,5}) \cdots (1 - r_{n-1,n}).
\end{align*}
$$

(A12)

Now, dividing $\det(\mathbf{C})$ for $(-1)^N \prod_{k=1}^{n} h_k g^N |\mathbf{M}_h|$ it is possible to yield an expression for the normalized determinant $\tilde{\mathbf{C}}$:

$$
\tilde{\mathbf{C}} = 1 - \sum_{k=1}^{n} F_1^2 + \Theta \simeq 1 - G^2,
$$

(A13)

where

$$
F_1^2 = \frac{u_1^2}{h_1 g (1 - r_{1,2})}, \quad F_2^2 = \frac{u_2^2 (1 - r_{k-1,k+1})}{h_k g (1 - r_{k-1,k}) (1 - r_{k,k+1})}
$$

(A14)
If we neglect the contribution of \( \tilde{\Theta} \), we get an expression like the two-layer formulation, where \( F_{k} \) is the Froude number of the \( k \)th layer and \( G \) is the composite Froude number.

The same expression of the Froude number can be found starting from the control condition indicated by Engqvist [1996] and Baines [1988]:

\[
C_{i,j} = \frac{F_{i}}{\gamma}, \quad \det(C) = 0, \quad (A15)
\]

where

\[
c = \begin{bmatrix}
\rho \frac{F_{i}^{2}}{\gamma} - \Delta \rho_{i} & -\rho \frac{F_{i}^{2}}{\gamma} + \rho \frac{F_{j}^{2}}{\gamma} - \Delta \rho_{j} & 0 & \cdots & 0 \\
0 & -\rho \frac{F_{i}^{2}}{\gamma} + \rho \frac{F_{j}^{2}}{\gamma} - \Delta \rho_{i} & \cdots & \cdots & \cdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & -\rho_{i} \frac{F_{i}^{2}}{\gamma} + \rho_{j} \frac{F_{j}^{2}}{\gamma} - \Delta \rho_{j} & -\rho_{i} \frac{F_{i}^{2}}{\gamma} + \rho_{j} \frac{F_{j}^{2}}{\gamma} - \Delta \rho_{j}
\end{bmatrix}
\]

\[
(A16)
\]

and \( F_{i}^{2} = \frac{u_{i}^{2}}{g} \frac{\Delta H_{i}}{\Delta \rho_{i}} \), \( \Delta \rho_{i} = (\rho_{i} - \rho_{i-1}) \), and \( h_{i,x} = \partial h_{i}/\partial x \). In this case, \( h_{i} \) represents the base height of the \( i \)th layer. Rendering explicit the system of equations (A14), we have

\[
\begin{align*}
[1] C_{11} h_{1,x} + C_{12} h_{1,x} & = 0 \\
[2] C_{21} h_{1,x} + C_{22} h_{1,x} + C_{23} h_{2,x} & = 0 \\
[3] C_{31} h_{1,x} + C_{32} h_{2,x} + C_{34} h_{3,x} & = 0 \\
[4] C_{41} h_{1,x} + C_{42} h_{2,x} + C_{43} h_{3,x} + C_{45} h_{4,x} & = 0 \\
[5] & \vdots \\
[n] C_{n-1,n-2,x} + C_{n,n-1,x} & = 0
\end{align*}
\]

It is possible to reduce the system of equations (A16) to \( n - 1 \), eliminating \( h_{0} \) in the second equation:

\[
[2] C_{22}^{*} h_{1,x} + C_{23}^{*} h_{2,x} = 0, \quad (A18)
\]

where

\[
C_{22}^{*} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \quad (A19)
\]

and

\[
C_{23}^{*} = C_{32} C_{11}. \quad (A20)
\]

If \( C_{22}^{*} = 0 \), there is a control condition that corresponds to the second layer stagnant [see Engqvist, 1996]. Replacing the \( C_{ij} \) terms, we yield

\[
C_{22}^{*} = \Delta \rho_{1} \Delta \rho_{2} - (\Delta \rho_{1} + \Delta \rho_{2}) p_{1} F_{1}^{2} - \Delta \rho_{1} p_{2} F_{1}^{2} + p_{1} F_{1}^{2} p_{2} F_{2}^{2} = 0, \quad (A21)
\]

that is,

\[
\frac{u_{1}^{2}}{g} \frac{\Delta H_{1}}{\Delta \rho_{1}} + \frac{u_{2}^{2}}{g} \frac{\Delta H_{2}}{\Delta \rho_{2}} + \frac{u_{1}^{2} u_{2}^{2}}{g^{2} \left( 1 - \frac{u_{1}}{u_{2}} \right)} \Delta H_{1} \Delta H_{2} = 1. \quad (A22)
\]

As the second layer is stagnant, the control condition is only on the first layer:

\[
F_{1}^{2} = \frac{u_{1}^{2}}{g} \frac{\Delta H_{1}}{\Delta \rho_{1}} = 1. \quad (A23)
\]

If \( C_{22}^{*} \neq 0 \), we get a \( n - 1 \) system of linear equations composed of the modified second relation equation \( C_{22}^{*} h_{1,x} + C_{23}^{*} h_{2,x} = 0 \) and the others \( 3 + n \) unmodified equations. If we also eliminate \( h_{1,x} \) from equation (A17), we get the modified relation

\[
[3] C_{22}^{*} h_{2,x} + C_{23}^{*} h_{3,x} = 0, \quad (A24)
\]

valid only if

\[
C_{33}^{*} = \left[ \begin{array}{c} C_{22}^{*} C_{23}^{*} C_{33} \end{array} \right] \neq 0. \quad (A25)
\]

If \( C_{33}^{*} = 0 \), there is a control condition such that the third layer is stagnant

\[
C_{33}^{*} = -\Delta \rho_{1} \Delta \rho_{2} - \Delta \rho_{1} p_{1} F_{1}^{2} - \Delta \rho_{1} p_{2} F_{2}^{2} + \Delta \rho_{1} p_{2} F_{1}^{2} + \Delta \rho_{1} p_{2} F_{2}^{2} + \Delta \rho_{2} p_{1} F_{1}^{2} + \Delta \rho_{2} p_{2} F_{2}^{2} + \Delta \rho_{2} p_{2} F_{2}^{2}
\]

\[
+ p_{1} F_{1}^{2} p_{2} F_{2}^{2} = 0, \quad (A26)
\]

that is,

\[
\frac{u_{1}^{2}}{g} \frac{\Delta H_{1}}{\Delta \rho_{1}} + \frac{u_{1}^{2}}{g} \frac{\Delta H_{1}}{\Delta \rho_{1}} + \frac{u_{1}^{2} u_{2}^{2}}{g^{2} \left( 1 - \frac{u_{1}}{u_{2}} \right)} \Delta H_{1} \Delta H_{2} = 1. \quad (A27)
\]

[57] Remembering that if this control is active, the third layer is stagnant, though the control condition is only on the first two layers, we can write

\[
G = \frac{u_{1}^{2}}{g} \frac{\Delta H_{1}}{\Delta \rho_{1}} + \frac{u_{2}^{2}}{g} \frac{\Delta H_{2}}{\Delta \rho_{2}} - \frac{u_{1}^{2} u_{2}^{2}}{g^{2} \left( 1 - \frac{u_{1}}{u_{2}} \right)} \Delta H_{1} \Delta H_{2} = 1, \quad (A28)
\]

with

\[
F_{1}^{2} = \frac{u_{1}^{2}}{g} \frac{\Delta H_{1}}{\Delta \rho_{1}} , \quad F_{2}^{2} = \frac{u_{2}^{2}}{g} \frac{\Delta H_{2}}{\Delta \rho_{2}} . \quad (A29)
\]

The iterative procedure can be continued to define deeper control and imply the same Froude number definition that we get previously from the direct calculation of determinant.
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